Cauchy's Proof of the Basel Problem

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Abstract

We prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

using Cauchy's elementary method. The argument combines a trigonometric inequality, De Moivre's formula, the Binomial Theorem, Vieta's formulas, and the Squeeze Theorem. No calculus beyond limits is required.

Overview

The proof has five steps:

1. A geometric inequality $\sin x < x < \tan x$ gives $\cot^2 x < \dfrac{1}{x^2} < \csc^2 x$.
2. De Moivre's formula and the Binomial Theorem express $\sin\big((2m+1)x\big)$ as a polynomial in $\cot^2 x$.
3. The numbers $\cot^2\left(\frac{k\pi}{2m+1}\right)$, $k = 1, \ldots, m$, are exactly the roots of that polynomial.
4. Vieta's formulas give the exact sum of those roots.
5. Summing the inequality from Step 1 and applying the Squeeze Theorem yields $\pi^2/6$.

1 The Trigonometric Squeeze

Lemma 1. *For* $0 < x < \frac{\pi}{2}$,

$$\sin x < x < \tan x.$$

Consequently,

$$\cot^2 x < \frac{1}{x^2} < \csc^2 x = 1 + \cot^2 x.$$

Proof. Place the angle x at the origin of the unit circle as in the figure. The triangle OAP lies inside the circular sector OAP, which in turn lies inside the right triangle OAT. Comparing areas,

$$\underbrace{\tfrac{1}{2}\sin x}_{\triangle OAP} \;<\; \underbrace{\tfrac{1}{2}x}_{\text{sector}} \;<\; \underbrace{\tfrac{1}{2}\tan x}_{\triangle OAT}.$$

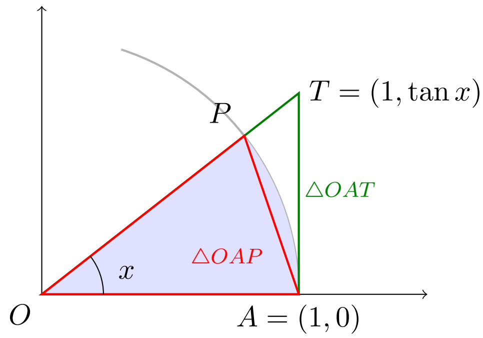

Figure 1: On the unit circle: triangle $OAP \subset$ sector $OAP \subset$ triangle OAT.

Multiplying by 2 gives $\sin x < x < \tan x$. Since all three quantities are positive, taking reciprocals reverses the inequalities:

$$\cot x < \frac{1}{x} < \csc x,$$

and squaring (all terms positive) gives $\cot^2 x < \dfrac{1}{x^2} < \csc^2 x$. Finally, $\csc^2 x = 1 + \cot^2 x$ follows from dividing $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$. $\square$

2 De Moivre's Formula and the Binomial Theorem

Let n be a positive integer and $0 < x < \frac{\pi}{2}$. By De Moivre's formula,

$$\cos(nx) + i\sin(nx) = (\cos x + i \sin x)^n.$$

Expanding the right side with the Binomial Theorem,

$$(\cos x + i\sin x)^n = \sum_{r=0}^{n} \binom{n}{r} \cos^{n-r} x\,(i \sin x)^r.$$

The imaginary terms are exactly those with r odd. Writing $r = 2j+1$ and using $i^{2j+1} = (-1)^j i$, we compare imaginary parts:

$$\sin(nx) = \sum_{j \geq 0} (-1)^j \binom{n}{2j+1} \cos^{n-2j-1} x\, \sin^{2j+1} x.$$

Now take $n = 2m+1$ to be odd. Then the sum runs over $j = 0, 1, \ldots, m$, and dividing both sides by $\sin^{2m+1} x$ (which is nonzero) gives

$$\frac{\sin\big((2m+1)x\big)}{\sin^{2m+1} x} = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \frac{\cos^{2m-2j} x}{\sin^{2m-2j} x} = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \big(\cot^2 x\big)^{m-j}. \tag{1}$$

Define the polynomial of degree m

$$P_m(t) = \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} t^{m-j} = \binom{2m+1}{1} t^m - \binom{2m+1}{3} t^{m-1} + \binom{2m+1}{5} t^{m-2} - \cdots.$$

Equation (1) says precisely that

$$\frac{\sin\big((2m+1)x\big)}{\sin^{2m+1} x} = P_m\big(\cot^2 x\big). \tag{2}$$

3 Locating the Roots

Lemma 2. *For $k = 1, 2, \ldots, m$, let*

$$x_k = \frac{k\pi}{2m+1}, \qquad t_k = \cot^2 x_k.$$

Then $t_1, \ldots, t_m$ are m distinct numbers, and they are exactly the roots of $P_m(t)$.

Proof. For $1 \le k \le m$ we have $0 < x_k < \frac{\pi}{2}$, so $\sin x_k \ne 0$. Also $(2m+1)x_k = k\pi$, so $\sin\big((2m+1)x_k\big) = 0$. By (2), $P_m(t_k) = 0$, so each t_k is a root.

On the interval $\left(0, \frac{\pi}{2}\right)$, the function $\cot x$ is positive and strictly decreasing, so $\cot^2 x$ is strictly decreasing as well. Since $x_1 < x_2 < \cdots < x_m$ all lie in this interval, $t_1 > t_2 > \cdots > t_m$, so the t_k are distinct.

A polynomial of degree m has at most m roots, so $t_1, \ldots, t_m$ are all of the roots of P_m. □

4 Vieta's Formulas

For a polynomial $a_m t^m + a_{m-1} t^{m-1} + \cdots + a_0$ with roots $t_1, \ldots, t_m$, Vieta's formulas give

$$t_1 + t_2 + \cdots + t_m = -\frac{a_{m-1}}{a_m}.$$

For P_m we have $a_m = \binom{2m+1}{1} = 2m+1$ and $a_{m-1} = -\binom{2m+1}{3}$. Therefore

$$\sum_{k=1}^{m} \cot^2 x_k = \frac{\binom{2m+1}{3}}{2m+1} = \frac{(2m+1)(2m)(2m-1)}{6\,(2m+1)} = \frac{m(2m-1)}{3}. \tag{3}$$

Using $\csc^2 x = 1 + \cot^2 x$,

$$\sum_{k=1}^{m} \csc^2 x_k = m + \frac{m(2m-1)}{3} = \frac{2m(m+1)}{3}. \tag{4}$$

5 The Squeeze

Theorem 1 (Basel Problem)**.**

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof. Apply Lemma 1 at each $x_k = \frac{k\pi}{2m+1}$, $k = 1, \ldots, m$:

$$\cot^2 x_k < \frac{(2m+1)^2}{k^2\pi^2} < \csc^2 x_k.$$

Summing over $k = 1, \ldots, m$ and using (3) and (4),

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)}{3}.$$

Multiplying through by $\dfrac{\pi^2}{(2m+1)^2}$,

$$\frac{\pi^2}{3} \cdot \frac{m(2m-1)}{(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{\pi^2}{3} \cdot \frac{2m(m+1)}{(2m+1)^2}. \tag{5}$$

Now let $m \to \infty$. Dividing numerator and denominator by m^2,

$$\lim_{m\to\infty} \frac{m(2m-1)}{(2m+1)^2} = \lim_{m\to\infty} \frac{2-\frac{1}{m}}{\left(2+\frac{1}{m}\right)^2} = \frac{2}{4} = \frac{1}{2}, \qquad \lim_{m\to\infty} \frac{2m(m+1)}{(2m+1)^2} = \lim_{m\to\infty} \frac{2+\frac{2}{m}}{\left(2+\frac{1}{m}\right)^2} = \frac{1}{2}.$$

So both the lower and upper bounds in (5) tend to $\dfrac{\pi^2}{3} \cdot \dfrac{1}{2} = \dfrac{\pi^2}{6}$. By the Squeeze Theorem,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

$\square$

Remark. The key idea is that the inequality $\cot^2 x < 1/x^2 < \csc^2 x$ traps $1/k^2$ between two quantities whose sums over the special angles $\frac{k\pi}{2m+1}$ can be computed *exactly*, thanks to De Moivre's formula and Vieta's formulas. The gap between the two sums is only m, which becomes negligible after dividing by $(2m+1)^2$.