Cardano's Method

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We are asked to solve the cubic equation

$$2x^3 + 4x^2 - 50x + 140 = 0$$

using Cardano's Method.

The first step is to make the cubic monotone - of the form $x^3 + ax^2 + bx + c = 0$ For the cubic equation above, this simply requires dividing each term by 2.

$$\frac{2}{2}x^3 + \frac{4}{2}x^2 - \frac{50}{2}x + \frac{140}{2} = 0$$

$$x^3 + 2x^2 - 25x + 70 = 0$$

We then transform this equation to make it a 'depressed cubic' by applying the transformation: $x = (y - \frac{a}{3})$. For our cubic equation above, the specific transformation is as follows:

$$x = \left(y - \frac{2}{3}\right)$$

Applying this transformation to the cubic equation above, we can complete the expansion and simplification to verify that our cubic is depressed, as it lacks an x^2 term.

$$\left(y - \frac{2}{3}\right)^3 + 2\left(y - \frac{2}{3}\right)^2 - 25\left(y - \frac{2}{3}\right) + 70 = 0$$

$$y^3 - 2y^2 + \frac{4}{3}y - \frac{8}{27} + 2\left(y^2 - \frac{4}{3}y + \frac{4}{9}\right) - 25\left(y - \frac{2}{3}\right) + 70 = 0$$

$$y^3 - 2y^2 + \frac{4}{3}y - \frac{8}{27} + 2y^2 - \frac{8}{3}y + \frac{8}{9} - 25y + \frac{50}{3} + 70 = 0$$

$$y^3 + \frac{4}{3}y - 25y - \frac{8}{3}y + \frac{8}{9} - \frac{8}{27} + \frac{50}{3} + 70 = 0$$

$$y^3 - \frac{79}{3}y + \frac{2356}{27} = 0$$

The next step to solving this equation relies on a second substitution. Let $y = (S + T)$

Then,

$$y^3 = (S + T)^3$$

$$y^3 = (S^3 + 3S^2T + 3ST^2 + T^3)$$

By rearranging,

$$y^3 = (S^3 + T^3) + (3S^2T + 3ST^2)$$

We can factor out a $(S + T)$ term like so

$$y^3 = (S^3 + T^3) + 3ST(S + T)$$

Then, we can apply the equality $y = (S+T)$ to substitute y back into the equation.

$$y^3 = (S^3 + T^3) + 3STy$$

Rearranging the equation, it becomes

$$y^3 - 3STy - (S^3 + T^3) = 0$$

Consider this equation and the general depressed cubic. Since the equations are of the same form, we can resolve S and T by equating the terms in our two equations.

$$y^3 + py + q = 0$$

$$y^3 - 3STy - (S^3 + T^3) = 0$$

The coefficient on the y term must be the same, which means

$$-3ST = p$$

The constant term must also be equal, which means

$$-(S^3 + T^3) = q$$

Using these two conditions, we can find S and T for our cubic $y^3 - \frac{79}{3}y + \frac{2356}{27} = 0$.

Since $p = -\frac{79}{3}$ and $q = \frac{2356}{27}$, S and T can be related like so

$$-3ST = -\frac{79}{3}$$

$$-(S^3 + T^3) = \frac{2356}{27}$$

Now, we can see the justification of breaking down y into two variables S and T. With the substitution, we have reduced the complexity to solving a system of equations.

Rearranging the first equation, we can make another substitution.

$$-3ST = -\frac{79}{3}$$

$$S = \frac{79}{9T}$$

Rearranging and substituting into our second equation,

$$-(S^3 + T^3) = \frac{2356}{27}$$

$$(\frac{79}{9T})^3 + T^3 = -\frac{2356}{27}$$

$$(\frac{493039}{729T^3}) + T^3 = -\frac{2356}{27}$$

The next step is to solve for T by rearranging the equation. Begin by multiplying both sides by T^3

$$(\frac{493039}{729T^3})(T^3) + (T^3)^2 = -(\frac{2356}{27})(T^3)$$

Rearranging yields

$$(T^3)^2 + \frac{2356}{27}T^3 + \frac{493039}{729} = 0$$

The problem can now be treated as a quadratic equation - of the form $ax^2 + bx + c = 0$. Instead of solving for x however, we can use the quadratic formula to solve for our term T^3.

Recall the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Applying this formula to our equation,

$$T^3 = \frac{-(\frac{2356}{27}) \pm \sqrt{(\frac{2356}{27})^2 - 4(1)(\frac{493039}{729})}}{2(1)}$$

We find two values for T^3.

$$T^3 \approx -8.598$$

$$T^3 \approx -78.66$$

Evaluating the cube root yields

$$T \approx -2.049$$

$$T \approx -4.285$$

The ambiguity of which value T is, is not a problem as we can assume that the other value will be S. Recalling that $y = (S + T)$, we can simply take the sum of these solutions to find y.

$$y = (S + T)$$

$$y \approx -6.334$$

From the very start, we made the substitution $x = (y - \frac{2}{3})$. As this equation still holds true, we can finally solve for x.

$$x \approx -6.334 - \frac{2}{3}$$

$$x = -7$$

We can verify that this solution works by substituting x back into our original cubic equation.

$$2x^3 + 4x^2 - 50x + 140 = 0$$

$$2(-7)^3 + 4(-7)^2 - 50(-7) + 140 = 0$$

$$-686 + 196 + 350 + 140 = 0$$

$$0 = 0$$

Thus, with Cardano's method, we've successfully solved our cubic equation $2x^3 + 4x^2 - 50x + 140 = 0$ finding that

$\boxed{\text{x = -7}}$